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2-uniform congruences in majority algebras and a closure operator

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Dedicated to the memory of Kazimierz Głazek

Abstract. A ternary term $m(x, y, z)$ of an algebra is called a *majority term* if the algebra satisfies the identities $m(x, x, y) = x$, $m(x, y, x) = x$ and $m(y, x, x) = x$. A congruence α of a finite algebra is called *uniform* if all of its blocks (i.e., classes) have the same number of elements. In particular, if all the α-blocks are two-element then α is said to be a 2-*uniform congruence*. If all congruences of A are uniform then A is said to be a *uniform algebra*. Answering a problem raised by Grätzer, Quackenbush and Schmidt [2], Kaarli [3] has recently proved that uniform finite lattices are congruence permutable.

In connection with Kaarli's result, our main theorem states that for every finite algebra A with a majority term any two 2-uniform congruences of A permute. Examples show that we can say neither "algebra" instead of "algebra with a majority term", nor "3-uniform" instead of "2-uniform".

Given two nonempty sets A and B, each relation $\rho \subseteq A \times B$ gives rise to a pair of closure operators, which are called the Galois closures on A and B induced by ρ. Galois closures play an important role in many parts of algebra, and they play the main role in formal concept analysis founded by Wille [4]. In order to prove our main theorem, we introduce a pair of smaller closure operators induced by ρ. These closure operators will hopefully find further applications in the future.

1. Introduction and main results

A ternary term $m(x, y, z)$ of an algebra is called a *majority term* if the algebra satisfies the identities $m(x, x, y) = x$, $m(x, y, x) = x$ and $m(y, x, x) = x$. Lattices are known to have majority terms: $m(x, y, z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$ and its dual. A congruence α of an algebra A is called a 2-*uniform congruence* if all the α-blocks (in other words, α-classes) are two-element. If all of its blocks have the same number of elements then α is called a *uniform congruence*. A finite algebra is said to be a *uniform algebra* if all of its congruences are uniform. A finite lattice

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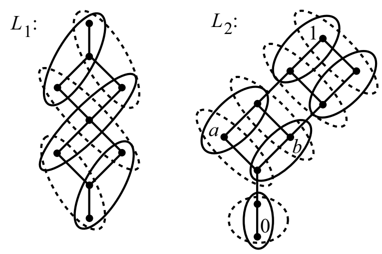


FIGURE 1

is called *isoform* if all blocks of any of its congruences are isomorphic lattices. Clearly, isoform lattices are uniform. Grätzer, Quackenbush and Schmidt [2] raised the question if finite isoform lattices are congruence permutable. Recently Kaarli [3] has shown even more: every finite uniform lattice is congruence permutable. Our main result includes algebras more general than lattices and does not assume that the algebra is uniform:

Theorem 1. *Let A be a finite algebra with a majority term. Then any two 2-uniform congruences of A permute.*

The lattice L_1 of Figure 1 with two non-permutable 3-uniform congruences, the one visualized by solid blocks and the other one by dotted blocks, indicates that Theorem 1 cannot be strengthened to 3-uniform congruences. Finiteness is also essential, for the lattice $(\mathbf{Z}; \max, \min)$ has exactly two 2-uniform congruences, but they do not permute. The projection kernels of the algebra

$$\{(a_0, b_0), (a_0, b_1), (a_1, b_1), (a_1, b_2), (a_2, b_2), (a_2, b_0)\}$$

with no operation indicate that 2-uniform congruences need not permute in *every* finite algebra. Notice that there are non-uniform finite lattices with 2-uniform congruences. For example, the lattice L_2 depicted in Figure 1 is not uniform, for its congruence with blocks $[0, a]$, $[b, 1]$ is not uniform. Yet, it has two 2-uniform congruences: one of them is depicted by solid blocks while the other with dotted blocks. Hence Theorem 1 adds something to Kaarli's result even for lattices, but this is not difficult to deduce from his proof. Even the following version can easily be extracted from his argument.

Proposition 1 (Implicit in Kaarli [3])**.** *Let B be a finite algebra with a ternary term m. Suppose that for each $n \in \{6, 7, \ldots, |B|\}$ the algebra has n-ary idempotent terms p_n and q_n such that the identities*

$$m(p_n(x_1, \ldots, x_n), x_i, q_n(x_1, \ldots, x_n)) = x_i,$$
$$m(x_i, p_n(x_1, \ldots, x_n), q_n(x_1, \ldots, x_n)) = x_i$$

hold in B for all $i \in \{1,\dots,n\}$. Then for any two uniform congruences α and β of B, if $\alpha \wedge \beta = 0_B$ then α and β permute.

Notice that for lattices we can choose $m(x,y,z) = (x \wedge y) \vee (x \wedge z) \vee (y \wedge z)$, $p_n(x_1,\dots,x_n) = x_1 \wedge \cdots \wedge x_n$ and $q_n(x_1,\dots,x_n) = x_1 \vee \cdots \vee x_n$. Note that the congruences of L_2 given in Figure 1 belong to the scope of of Theorem 1 but not to the scope of Proposition 1.

The reason that Proposition 1 occurs here is twofold: to make the present paper self-contained, and to illuminate a connection between majority terms and uniform congruences. The question if there are algebras without lattice reduct that satisfy the conditions of Proposition 1 has not been investigated.

Now, as a crucial step towards Theorem 1, we are going to define some new closure operators which, in general, are smaller than those induced by Galois connections. The importance of these closure operators in knowledge systems (concept lattices, association rules) and algebra is outlined in [1]. Following Wille's terminology from [4], a triplet

$$(A^{(0)}, A^{(1)}, \rho)$$

is called a *context* if $A^{(0)}$ and $A^{(1)}$ are nonempty sets and $\rho \subseteq A^{(0)} \times A^{(1)}$, a binary relation. From what follows, we fix a context $(A^{(0)}, A^{(1)}, \rho)$ and let $\rho_0 = \rho$ and $\rho_1 = \rho^{-1}$. From now on, unless otherwise stated, i will be an arbitrary element of $\{0,1\}$. So whatever we say including i without specification, it will be understood as prefixed by $\forall i$.

The set of all subsets of $A^{(i)}$ will be denoted by $P(A^{(i)})$.

It is often, especially in the finite case, convenient to depict our context in the usual form: a binary table with column labels from $A^{(0)}$, row labels from $A^{(1)}$ and a cross in the intersection of the x-th column and y-th row iff $(x,y) \in \rho$. (A concrete example will be given later, in Figure 2.) We will refer to this table as the *context table*, and we will use it to visualize some of the notions we introduce.

A mapping $\mathcal{D}^{(i)} \colon P(A^{(i)}) \to P(A^{(i)})$ is called a *closure operator* if it is *extensive* (i.e., $X \subseteq \mathcal{D}^{(i)}(X)$ for all $X \in P(A^{(i)})$), *monotone* (i.e., $X \subseteq Y \implies \mathcal{D}^{(i)}(X) \subseteq \mathcal{D}^{(i)}(Y)$), and *idempotent* (i.e., $\mathcal{D}^{(i)}(\mathcal{D}^{(i)}(X)) = \mathcal{D}^{(i)}(X)$ for all $X \in P(A^{(i)})$. By a *pair of closure operators* we always mean a pair $\mathcal{D} = (\mathcal{D}^{(0)}, \mathcal{D}^{(1)})$ where $\mathcal{D}^{(0)} \colon P(A^{(0)}) \to P(A^{(0)})$ and $\mathcal{D}^{(1)} \colon P(A^{(1)}) \to P(A^{(1)})$ are closure operators. If $\mathcal{E} = (\mathcal{E}^{(0}, \mathcal{E}^{(1)})$ is another pair of closure operators then $\mathcal{D} \subseteq \mathcal{E}$ means that $\mathcal{D}^{(0)}(X) \subseteq \mathcal{E}^{(0)}(X)$ and $\mathcal{D}^{(1)}(Y) \subseteq \mathcal{E}^{(1)}(Y)$ for all $X \in P(A^{(0)})$ and $Y \in P(A^{(1)})$.

Associated with $(A^{(0)}, A^{(1)}, \rho)$, we define a sequence $\mathcal{C}_n = (\mathcal{C}_n^{(0)}, \mathcal{C}_n^{(1)})$ of pairs of closure operators. (It will be shown later that the $\mathcal{C}_n^{(i)}$ we are defining are indeed closure operators.) For $X \in P(A^{(i)})$ let

$$X\rho_i = \{y \in A^{(1-i)} : \text{for all } x \in X,\ (x,y) \in \rho_i\},$$

and for $X \in P(A^{(i)})$, define

$$\mathcal{C}_0^{(i)}(X) := (X\rho_i)\rho_{1-i} = \bigcap_{y\in X\rho_i} (\{y\}\rho_{1-i}) \ .$$

So, $\mathcal{C}_0 = (\mathcal{C}_0^{(0)}, \mathcal{C}_0^{(1)})$ is the well-known pair of Galois closure operators.

If $\mathcal{C}_n$ is already defined then let

$$X\psi_i := \{Y \in P(A^{(1-i)}) : \text{there is a surjection } \varphi\colon X \to Y \text{ such that } (x, x\varphi) \in \rho_i \text{ for all } x \in X\},$$

and set

$$\mathcal{C}_{n+1}^{(i)}(X) := \mathcal{C}_n^{(i)}(X) \cap \bigcap_{Y \in X\psi_i} \ \bigcup_{y \in \mathcal{C}_n^{(1-i)}(Y)} \{y\}\rho_{1-i} \ . \tag{1}$$

Finally, throughout the paper, let

$$\mathcal{C} = (\mathcal{C}^{(0)}, \mathcal{C}^{(1)}) := (\bigcap_{n=0}^{\infty} \mathcal{C}_n^{(0)}, \bigcap_{n=0}^{\infty} \mathcal{C}_n^{(1)}),$$

which means that, for all $X \in P(A^{(i)})$,

$$\mathcal{C}^{(i)}(X) = \bigcap_{n=0}^{\infty} \mathcal{C}_n^{(i)}(X).$$

Lemma 1. *$\mathcal{C}$ and $\mathcal{C}_n$, $n = 0, 1, \dots,$ are pairs of closure operators. Further,*

$$\mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \mathcal{C}_2 \supseteq \dots \supseteq \mathcal{C} \, .$$

From now on we assume that $(A^{(0)}, A^{(1)}, \rho)$ is finite. We say that $(A^{(0)}, A^{(1)}, \rho)$ is a *subdirect context* if the natural projections maps

$$\pi_i \colon \rho \to A^{(i)}, \quad (x_0, x_1) \mapsto x_i$$

are surjective. This means that the context table contains neither an empty column nor an empty row. This notion corresponds to that of subdirect products. If ρ happens to be a subalgebra of the direct product $A^{(0)} \times A^{(1)}$ of algebras $A^{(0)}$ and $A^{(1)}$ then ρ is a subdirect product of $A^{(0)}$ and $A^{(1)}$ iff $(A^{(0)}, A^{(1)}, \rho)$ is a subdirect context. When $A^{(0)}$ and $A^{(1)}$ are algebras then $\rho \leq_{\text{sd}} A^{(0)} \times A^{(1)}$ will denote that ρ is subdirect product of $A^{(0)}$ and $A^{(1)}$.

Our $(A^{(0)}, A^{(1)}, \rho)$ will be said to be a *decomposable context* if there are nonempty sets $B^{(i)}$ and $C^{(i)}$ with $B^{(i)} \cup C^{(i)} = A^{(i)}$ and $B^{(i)} \cap C^{(i)} = \emptyset$ such that

$$\rho = (\rho \cap (B^{(0)} \times B^{(1)})) \cup (\rho \cap (C^{(0)} \times C^{(1)})).$$

Otherwise $(A^{(0)}, A^{(1)}, \rho)$ is called an *indecomposable context.* If $(A^{(0)}, A^{(1)}, \rho)$ is subdirect then it is indecomposable iff the join of the kernels of the natural projection maps is the largest equivalence on ρ, i.e., $\ker(\pi_0) \vee \ker(\pi_1) = 1_\rho$. Therefore indecomposability in the subdirect case means that from any cross in the context table we can reach any other cross by vertical and horizontal moves from cross to cross.

We say that the context is *uniform* if $|\{x\}\rho_i| = |\{y\}\rho_i|$ for all $x, y \in A^{(i)}$. In the terminology of context tables, any two columns contain the same number of crosses and any two rows contain the same number of crosses. The context is 2-uniform if $|\{x\}\rho_i| = 2$ for all $x \in A^{(i)}$. This means that any row and any column contains exactly two crosses. Notice that if $(A^{(0)}, A^{(1)}, \rho)$ is 2-uniform then $|A^{(0)}| = |A^{(1)}|$.

We say that $(A^{(0)}, A^{(1)}, \rho)$ contains *a nondegenerate triangle* iff there is an $i \in \{0,1\}$ and there are $a, b, c \in A^{(i)}$ such that

$$\mathcal{C}^{(i)}(\{a,b\}) \cap \mathcal{C}^{(i)}(\{a,c\}) \cap \mathcal{C}^{(i)}(\{b,c\}) = \emptyset \ .$$

Notice that in this case $|\{a,b,c\}| = 3$.

The key role in the proof of Theorem 1 will be played by the following statement.

Proposition 2. *If $(A^{(0)}, A^{(1)}, \rho)$ is a finite 2-uniform indecomposable subdirect context with $|A^{(i)}| \geq 3$, then it contains a nondegenerate triangle.*

2. Proofs and further statements

Proof of Proposition 1. This argument is basically from [3], tailored to peculiarities of the present paper. Clearly, m is an idempotent term. We can assume that B is an idempotent algebra, for otherwise we can replace it by its full idempotent reduct. Suppose α and β are uniform congruences of B with $\alpha \wedge \beta = 0_B$. It suffices to show that for any block A of $\alpha \vee \beta$, the restriction of α and that of β to A permute. But A is a subalgebra. So, instead of B, it is sufficient to deal only with A where $\alpha \vee \beta = 1_A$ and $\alpha \wedge \beta = 0_A$. We may assume that $|A| \geq 6$, for otherwise any two uniform congruences (or equivalences) permute.

Consider the canonical embedding that maps A onto $A' \leq_{\mathrm{sd}} A/\alpha \times A/\beta$, and adopt the notations $\rho := A'$, $A^{(0)} := A/\alpha$ and $A^{(1)} := A/\beta$. Then $(A^{(0)}, A^{(1)}, \rho)$ is a uniform indecomposable subdirect context. It suffices to show that $\rho = A^{(0)} \times A^{(1)}$, which will clearly imply that α and β permute.

Let $\rho = \{(a_1^{(0)}, a_1^{(1)}), \ldots, (a_n^{(0)}, a_n^{(1)})\}$. Here $A^{(i)} = \{a_1^{(i)}, \ldots, a_n^{(i)}\}$. Note that $n = |\rho| = |A| \geq 6$ but $|A^{(i)}|$ need not be equal to n. Define

$$\begin{aligned} u = (u^{(0)}, u^{(1)}) &:= (p_n(a_1^{(0)}, \ldots, a_n^{(0)}), p_n(a_1^{(1)}, \ldots, a_n^{(1)})) \\ &= p_n((a_1^{(0)}, a_1^{(1)}), \ldots, (a_n^{(0)}, a_n^{(1)})) \in \rho, \\ v = (v^{(0)}, v^{(1)}) &:= (q_n(a_1^{(0)}, \ldots, a_n^{(0)}), q_n(a_1^{(1)}, \ldots, a_n^{(1)})) \\ &= q_n((a_1^{(0)}, a_1^{(1)}), \ldots, (a_n^{(0)}, a_n^{(1)})) \in \rho. \end{aligned}$$

Now let

$$\{(u^{(0)}, a^{(1)}_{i_1}), \dots, (u^{(0)}, a^{(1)}_{i_k})\} \quad \text{(with exactly } k \text{ elements)}$$

be the $\ker(\pi_0)$-block of u, and let

$$\{(a^{(0)}_{j_1}, u^{(1)}), \dots, (a^{(0)}_{j_\ell}, u^{(1)})\} \quad \text{(with exactly } \ell \text{ elements)}$$

be the $\ker(\pi_1)$-block of u. Using the identities, we obtain for $1 \le r \le k$ and $1 \le s \le \ell$ that

$$\begin{aligned}(a^{(0)}_{j_s}, a^{(1)}_{i_r}) &= (m(u^{(0)}, a^{(0)}_{j_s}, v^{(0)}), m(a^{(1)}_{i_r}, u^{(1)}, v^{(1)})) \\ &= m((u^{(0)}, a^{(1)}_{i_r}), (a^{(0)}_{j_s}, u^{(1)}), (v^{(0)}, v^{(1)})) \in \rho.\end{aligned}$$

This implies that

$$X_r := \{(a^{(0)}_{j_1}, a^{(1)}_{i_r}), \dots, (a^{(0)}_{j_\ell}, a^{(1)}_{i_r})\} \subseteq \ker(\pi_1).$$

By uniformity, all $\ker(\pi_1)$-blocks are of size ℓ, whence X_r is a $\ker(\pi_1)$-block for all $r \in \{1, \dots, k\}$. Similarly, we obtain that

$$Y_s := \{(a^{(0)}_{j_s}, a^{(1)}_{i_1}), \dots, (a^{(0)}_{j_s}, a^{(1)}_{i_k})\} \subseteq \ker(\pi_0)$$

is a $\ker(\pi_0)$-block for all $s \in \{1, \dots, \ell\}$. Hence

$$\kappa := \{a^{(1)}_{j_1}, \dots, a^{(1)}_{j_\ell}\} \times \{a^{(1)}_{i_1}, \dots, a^{(1)}_{i_k}\} \subseteq \rho$$

and κ is closed with respect to $\ker(\pi_0) \vee \ker(\pi_1)$. Since the context is indecomposable, we conclude $\kappa = \rho$, $A^{(0)} = \{a^{(0)}_{j_1}, \dots, a^{(0)}_{j_\ell}\}$ and $A^{(1)} = \{a^{(1)}_{i_1}, \dots, a^{(1)}_{i_k}\}$. Hence $\rho = A^{(0)} \times A^{(1)}$. □

Proof of Lemma 1. It is well known that $\mathcal{C}_0$ is a pair of closure operators. Suppose that $\mathcal{C}_n$ is a pair of closure operators.

Let $X \subseteq U \in P(A^{(i)})$ and let u belong to $\mathcal{C}^{(i)}_{n+1}(X)$, i.e., to the right-hand side of (1). Since $\mathcal{C}^{(i)}_n(X) \subseteq \mathcal{C}^{(i)}_n(U)$, it suffices to show that u belongs to the intersection in

$$\mathcal{C}^{(i)}_{n+1}(U) = \mathcal{C}^{(i)}_n(U) \cap \bigcap_{V \in U\psi_i} \; \bigcup_{y \in \mathcal{C}^{(1-i)}_n(V)} \{y\}\rho_{1-i} \; . \tag{2}$$

Let V be an arbitrary member of $U\psi_i$ by means of a surjection $\varphi\colon U \to V$ with $(x, x\varphi) \in \rho_i$ for all $x \in U$. Then $Y := X\varphi|_X$ is clearly in $X\psi_i$, and $Y \subseteq U\varphi = V$. Since $\mathcal{C}^{(1-i)}_n(Y) \subseteq \mathcal{C}^{(1-i)}_n(V)$ by the induction hypothesis,

$$u \in \bigcup_{y \in \mathcal{C}^{(1-i)}_n(Y)} \{y\}\rho_{1-i} \subseteq \bigcup_{y \in \mathcal{C}^{(1-i)}_n(V)} \{y\}\rho_{1-i}.$$

This shows that $u \in \mathcal{C}^{(i)}_{n+1}(U)$, whence $\mathcal{C}^{(i)}_{n+1}$ is monotone.

Now let $z \in X \in P(A^{(i)})$ and let $Y \in X\psi_i$ by means of a surjection $\varphi\colon X \to Y$ with $(x, x\varphi) \in \rho_i$ for all $x \in X$. In particular for $y := z\varphi$ we have $(y, z) \in \rho_{1-i}$, i.e., $z \in \{y\}\rho_{1-i}$. Since $\mathcal{C}_n^{(1-i)}$ is extensive, $y \in Y \subseteq \mathcal{C}_n^{(1-i)}(Y)$ shows that this y actually occurs in the right-hand side of (1). Therefore, from $z \in \{y\}\rho_{1-i}$ and $z \in X \subseteq \mathcal{C}_n^{(i)}(X)$ we obtain $x \in \mathcal{C}_{n+1}^{(i)}(X)$, showing that $\mathcal{C}_{n+1}^{(i)}$ is extensive.

Now, to show that $\mathcal{C}_{n+1}^{(i)}$ is idempotent, let $X \in P(A^{(i)})$, $U = \mathcal{C}_{n+1}^{(i)}(X)$ and $v \in \mathcal{C}_{n+1}^{(i)}(U)$. We need to show that $v \in U$. The easy part is as follows:

$$v \in \mathcal{C}_{n+1}^{(i)}(U) = \mathcal{C}_{n+1}^{(i)}(\mathcal{C}_{n+1}^{(i)}(X)) \subseteq \mathcal{C}_{n+1}^{(i)}(\mathcal{C}_n^{(i)}(X)) \subseteq \mathcal{C}_n^{(i)}(\mathcal{C}_n^{(i)}(X)) = \mathcal{C}_n^{(i)}(X).$$

To deal with the other part of the right-hand side of (1), let Y be an arbitrary member of $\in X\psi_i$ by means of a surjection $\varphi\colon X \to Y$ with $(x, x\varphi) \in \rho_i$ for all $x \in X$. We know from (1), which determines U, that for each $z \in U \setminus X$ we can choose an element $y_z \in \mathcal{C}_n^{(1-i)}(Y)$ with $z \in \{y_z\}\rho_{1-i}$, i.e., $(z, y_z) \in \rho_i$. We define a map

$$\mu\colon U \to \mathcal{C}_n^{(1-i)}(Y), \quad z \mapsto \begin{cases} z\varphi \text{ if } z \in X \\ y_z \text{ if } z \in U \setminus X\ . \end{cases}$$

Let $V := U\mu \subseteq \mathcal{C}_n^{(1-i)}(Y)$. Clearly, $V \in U\psi_i$, so V takes part in (2). Hence

$$v \in \bigcup_{y \in \mathcal{C}_n^{(1-i)}(V)} \{y\}\rho_{1-i}\,.$$

So $v \in \{y\}\rho_{1-i}$ for a suitable $y \in \mathcal{C}_n^{(1-i)}(V)$. Using the induction hypothesis we obtain $y \in \mathcal{C}_n^{(1-i)}(V) \subseteq \mathcal{C}_n^{(1-i)}(\mathcal{C}_n^{(1-i)}(Y)) = \mathcal{C}_n^{(1-i)}(Y)$. Thus, after establishing for an arbitrary $Y \in X\psi_i$ that $v \in \{y\}\rho_{1-i}$ for a suitable $y \in \mathcal{C}_n^{(1-i)}(Y)$, we have shown that $\mathcal{C}_{n+1}^{(i)}$ is idempotent, and so it is a closure operator.

Finally, $\mathcal{C}^{(i)}$ is clearly extensive and monotone. For any $n \in \mathbf{N}_0$ and $X \in P(A^{(i)})$,

$$\mathcal{C}^{(i)}(\mathcal{C}^{(i)}(X)) \subseteq \mathcal{C}^{(i)}(\mathcal{C}_n^{(i)}(X)) \subseteq \mathcal{C}_n^{(i)}(\mathcal{C}_n^{(i)}(X)) = \mathcal{C}_n^{(i)}(X),$$

which gives $\mathcal{C}^{(i)}(\mathcal{C}^{(i)}(X)) \subseteq \mathcal{C}^{(i)}(X)$. Therefore $\mathcal{C}$ is a pair of closure operators. $\square$

Now we establish a connection between contexts and majority terms. In fact, we do this in a slightly more general form than needed. For $k \geq 3$, by a k-ary *near-unanimity term* we mean a k-ary term m satisfying the identities

$$m(y,x,x,\ldots,x) = x,\ m(x,y,x,\ldots,x) = x,\ m(x,x,y,\ldots,x) = x, \ldots$$
$$\ldots, m(x,x,x,\ldots,y) = x.$$

Ternary near-unanimity terms are exactly the majority terms.

Proposition 3. *Suppose $(A^{(0)}, A^{(1)}, \rho)$ is also a subdirect product $\rho \leq_{sd} A^{(0)} \times A^{(1)}$ such that the algebra ρ, and therefore its homomorphic images $A^{(0)}$ and $A^{(1)}$, have* *a k-ary near-unanimity term m. Then for any $a_1^{(i)}, \dots, a_k^{(i)} \in A^{(i)}$ we have*

$$m(a_1^{(i)}, \dots, a_k^{(i)}) \in \mathcal{C}^{(i)}(\{a_2^{(i)}, \dots, a_k^{(i)}\}) \cap \mathcal{C}^{(i)}(\{a_1^{(i)}, a_3^{(i)}, \dots, a_k^{(i)}\}) \\ \cap \mathcal{C}^{(i)}(\{a_1^{(i)}, a_2^{(i)}, a_4^{(i)}, \dots, a_k^{(i)}\}) \cap \dots \cap \mathcal{C}^{(i)}(\{a_1^{(i)}, \dots, a_{k-1}^{(i)}\}).$$

Proof. By symmetry, it suffices to show that

$$m(a_1^{(j)}, \dots, a_k^{(j)}) \in \mathcal{C}_n^{(j)}(\{a_1^{(j)}, \dots, a_{k-1}^{(j)}\}) \tag{3}$$

for all $j \in \{0, 1\}$ and $n \in \mathbf{N}_0$. This will be done via induction on n. Let $u = m(a_1^{(i)}, \dots, a_k^{(i)})$ and $X = \{a_1^{(i)}, \dots, a_{k-1}^{(i)}\}$.

To prove $u \in \mathcal{C}_0^{(i)}(X) = (X\rho_i)\rho_{1-i}$, suppose $y \in X\rho_i$. Then $(a_1^{(i)}, y)$, $\dots, (a_{k-1}^{(i)}, y) \in \rho_i$. Since $\rho_i \leq_{\mathrm{sd}} A^{(i)} \times A^{(1-i)}$ is a subdirect product, there exists a $z \in A^{(1-i)}$ with $(a_k^{(i)}, z) \in \rho_i$. It follows that

$$(u, y) = (m(a_1^{(i)}, \dots, a_k^{(i)}), m(y, \dots, y, z)) = m((a_1^{(i)}, y), \dots, (a_{k-1}^{(i)}, y), (a_k^{(i)}, z)) \in \rho_i\,,$$

i.e., $u \in \{y\}\rho_{1-i}$. Hence $u \in \mathcal{C}_0^{(i)}(X)$, indeed.

Now suppose that (3) holds for $n \in \mathbf{N}_0$. Then $u \in \mathcal{C}_n^{(i)}(X)$. To show that u belongs to the intersection in (1), consider an arbitrary $Y \in X\psi_i$ together with a surjection $\varphi\colon X \to Y$, $a_1^{(i)} \mapsto a_1^{(1-i)}, \dots, a_{k-1}^{(i)} \mapsto a_{k-1}^{(1-i)}$ such that

$$(a_1^{(i)}, a_1^{(1-i)}), \dots, (a_{k-1}^{(i)}, a_{k-1}^{(1-i)}) \in \rho_i.$$

Here $\{a_1^{(1-i)}, \dots, a_{k-1}^{(1-i)}\} = Y$. Since the context is subdirect, there exists a $c \in A^{(1-i)}$ with $(a_k^{(i)}, c) \in \rho_i$. Define $y := m(a_1^{(1-i)}, \dots, a_{k-1}^{(1-i)}, c)$. The induction hypothesis yields

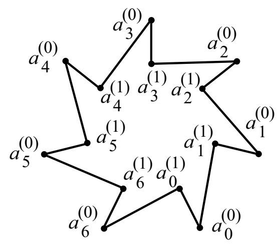
$$y \in \mathcal{C}_n^{(1-i)}(\{a_1^{(1-i)}, \dots, a_{k-1}^{(1-i)}\}) = \mathcal{C}_n^{(1-i)}(Y).$$

Now

$$(u, y) = (m(a_1^{(i)}, \dots, a_k^{(i)}), m(a_1^{(1-i)}, \dots, a_{k-1}^{(1-i)}, c)) \\ = m((a_1^{(i)}, a_1^{(1-i)}), \dots, (a_{k-1}^{(i)}, a_{k-1}^{(1-i)}), (a_k^{(i)}, c)) \in \rho_i$$

i.e., $u \in \{y\}\rho_{1-i}$. Therefore $u \in \mathcal{C}_{n+1}^{(i)}(X)$. □

Proof of Proposition 2. Consider the bipartite graph $(A^{(0)} \dot\cup A^{(1)}, \rho_0 \cup \rho_1)$. (Here $\dot\cup$ indicates the assumption $A^{(0)} \cap A^{(1)} = \emptyset$, which does not hurt generality.) This is a 2-regular graph (every vertex has degree 2), and it is connected since $(A^{(0)}, A^{(1)}, \rho)$ is indecomposable. Hence, as it is easy to see, it contains a Hamiltonian cycle. Due



	$a_0^{(0)}$	$a_1^{(0)}$	$a_2^{(0)}$	$a_3^{(0)}$	$a_4^{(0)}$	$a_5^{(0)}$	$a_6^{(0)}$
$a_0^{(1)}$	×						×
$a_1^{(1)}$	×	×					
$a_2^{(1)}$		×	×				
$a_3^{(1)}$			×	×			
$a_4^{(1)}$				×	×		
$a_5^{(1)}$					×	×	
$a_6^{(1)}$						×	×

FIGURE 2

to this cycle, we may assume that the vertices are indexed in the following way:

$$A^{(0)} = \{a_0^{(0)}, a_1^{(0)}, \dots, a_{k-1}^{(0)}\}, \quad A^{(1)} = \{a_0^{(1)}, a_1^{(1)}, \dots, a_{k-1}^{(1)}\},$$
$$\rho = \bigcup_{j \in \mathbf{Z}_k} \{(a_j^{(0)}, a_j^{(1)}),\ (a_j^{(0)}, a_{j+1}^{(1)})\}.$$

Here and in the rest of the proof, all index additions and subtractions are understood in $\mathbf{Z}_k$, i.e., modulo k. For $k = 7$ the situation is depicted in Figure 2. The subsets $\{a_j^{(i)}, a_{j+1}^{(i)}, \dots, a_{j+t-1}^{(i)}\}$ of $A^{(i)}$ will be called *arcs* of length t or size t, $0 \le t \le k$. We claim that for all arcs $\{a_j^{(i)}, a_{j+1}^{(i)}, \dots, a_{j+t-1}^{(i)}\}$ of $A^{(i)}$,

$$\mathcal{C}^{(i)}(\{a_j^{(i)}, a_{j+1}^{(i)}, \dots, a_{j+t-1}^{(i)}\}) = \{a_j^{(i)}, a_{j+1}^{(i)}, \dots, a_{j+t-1}^{(i)}\}\,. \tag{4}$$

(To make a comparison with the Galois closure we notice that $\mathcal{C}_0^{(i)}(X) = A^{(i)}$ holds in Figure 2 whenever X is an arc with $|X| \ge 3$.)

We prove (4) via induction on t. The case $t = 0$ is evident by $|A^{(i)}| \ge 3$. If $t = 1$ then

$$\mathcal{C}^{(0)}(\{a_j^{(0)}\}) \subseteq \mathcal{C}_0^{(0)}(\{a_j^{(0)}\}) = (\{a_j^{(0)}\}\rho_0)\rho_1 = \{a_j^{(1)}, a_{j+1}^{(1)}\}\rho_1 = \{a_j^{(0)}\}$$

yields $\mathcal{C}^{(0)}(\{a_j^{(0)}\}) = \{a_j^{(0)}\}$, and $\mathcal{C}^{(1)}(\{a_j^{(1)}\}) = \{a_j^{(1)}\}$ follows similarly.

Now let $X = \{a_j^{(0)}, a_{j+1}^{(0)}, \dots, a_{j+t}^{(0)}\}$ be an arc with length $t+1 \ge 2$. Observe that, with the notations of (1), the surjection $a_j^{(0)} \mapsto a_{j+1}^{(1)}$, $a_{j+1}^{(0)} \mapsto a_{j+1}^{(1)}$, $a_{j+2}^{(0)} \mapsto a_{j+2}^{(1)}$, $\dots$, $a_{j+t}^{(0)} \mapsto a_{j+t}^{(1)}$ establishes that $Y := \{a_{j+1}^{(1)}, a_{j+2}^{(1)}, \dots, a_t^{(1)}\} \in X\psi_0$. Because of finiteness we can choose an n such that $\mathcal{C} = \mathcal{C}_n$. The induction hypothesis gives $\mathcal{C}_n^{(1-0)}(Y) = \mathcal{C}^{(1-0)}(Y) = Y$. This and the peculiarities of the context imply that, for this Y,

$$\bigcup_{y \in \mathcal{C}_n^{(1-0)}(Y)} \{y\}\rho_{1-0} \subseteq X,$$

and we conclude $\mathcal{C}^{(0)}(X) = \mathcal{C}^{(0)}_{n+1}(X) = X$. The case when $X \subseteq A^{(1)}$ is similar.

Now, armed with (4) we can show that $\mathcal{C}$ is trivial, i.e., $\mathcal{C}^{(i)}(X) = X$ holds for any $X \in P(A^{(i)})$. Indeed, using that $\mathcal{C}^{(i)}$ is monotone and $A^{(i)} \setminus \{a^{(i)}_j\}$ is always an arc, we obtain

$$\begin{aligned}\mathcal{C}^{(i)}(X) = \mathcal{C}^{(i)}\Big(\bigcap_{a^{(i)}_j \notin X} (A^{(i)} \setminus \{a^{(i)}_j\})\Big) &\subseteq \bigcap_{a^{(i)}_j \notin X} \mathcal{C}^{(i)}(A^{(i)} \setminus \{a^{(i)}_j\}) \\ &= \bigcap_{a^{(i)}_j \notin X} (A^{(i)} \setminus \{a^{(i)}_j\}) = X.\end{aligned}$$

We have proved that $\mathcal{C}$ is trivial. Thus, any triangle of three distinct elements is nondegenerate. $\square$

Proof of Theorem 1. Let α_0 and α_1 be 2-uniform congruences of A, where A has a majority term $m(x, y, z)$. Since α_0 and α_1 are congruences of the algebra (A, m), we may assume that A is the algebra (A, m). It suffices to show that the restrictions of α_0 and α_1 to an arbitrary block of $\alpha_0 \vee \alpha_1$ permute. Since congruence blocks are subalgebras by the idempotency of m, we may assume that A is an $\alpha_0 \vee \alpha_1$-block, i.e., $\alpha_0 \vee \alpha_1 = 1_A$. If $|A| \le 4$ then any two of its 2-uniform congruences permute and there is nothing to prove.

Hence, seeking a contradiction, we assume that $|A| \ge 5$. Then 2-uniformity of α_0 and α_1 together with $\alpha_0 \vee \alpha_1 = 1_A$ gives $\alpha_0 \wedge \alpha_1 = 0_A$. Let $A^{(i)} := A/\alpha_i$. Clearly, $|A^{(i)}| \ge 3$. Consider the canonical embedding that maps A onto $\rho \le_{\mathrm{sd}} A^{(0)} \times A^{(1)}$. Since the projection kernels of this subdirect product correspond to α_0 and α_1, $(A^{(0)}, A^{(1)}, \rho)$ is a 2-uniform indecomposable subdirect context. By Proposition 2 there is a nondegenerate triangle $\{a, b, c\} \subseteq A^{(i)}$ for some $i \in \{0, 1\}$. Applying Proposition 3 we obtain

$$m(a, b, c) \in \mathcal{C}^{(i)}(\{a, b\}) \cap \mathcal{C}^{(i)}(\{a, c\}) \cap \mathcal{C}^{(i)}(\{b, c\}) = \emptyset,$$

a contradiction. $\square$

3. Concluding remarks

Problem 1. How far can Kaarli's theorem be extended beyond lattices?

After examining thousands of contexts we formulate the following

Conjecture. *Every finite uniform indecomposable subdirect context* $(A^{(0)}, A^{(1)}, \rho)$ *with* $|A^{(i)}| \ge 3$ *and* $\rho \ne A^{(0)} \times A^{(1)}$ *has a nondegenerate triangle.*

If this conjecture was proved, the following partial answer to Problem 1 would follow: "finite uniform algebras with majority term are congruence permutable".

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